Soundness of propositional logic

The natural deduction rules make it possible for us to develop rigorous threads of argumentation, in the course of which we arrive at a conclusion ψ assuming certain other propositions $\varphi 1$, $\varphi 2,...,\varphi n$. In that case, we said that the sequent $\varphi 1$, $\varphi 2,...,\varphi n$ ψ is valid. Do we have any evidence that these rules are all correct in the sense that valid sequents all 'preserve truth' computed by our truth-table semantics?

Given a proof of $\varphi 1$, $\varphi 2,...,\varphi n$ ψ, is it conceivable that there is a valuation in which ψ above is false although all propositions $\varphi 1$, $\varphi 2,...,\varphi n$ are true? Fortunately, this is not the case and in this subsection we demonstrate why this is so. Let us suppose that some proof in our natural deduction calculus has established that the sequent $\varphi 1$, $\varphi 2,...,\varphi n$ ψ is valid. We need to show: for all valuations in which all propositions $\varphi 1$, $\varphi 2,...,\varphi n$ evaluate to T, ψ evaluates to T as well.

Theorem 1.35 (Soundness) *Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional logic formulas. If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.*

PROOF: Since $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid we know there is a proof of ψ from the premises $\phi_1, \phi_2, \ldots, \phi_n$. We now do a pretty slick thing, namely, we reason by *mathematical induction on the length of this proof!* The length of a proof is just the number of lines it involves. So let us be perfectly clear about what it is we mean to show. We intend to show the assertion $M(k)$:

> *'For all sequents $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ $(n \geq 0)$ which have a proof of length k, it is the case that $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.'*

by course-of-values induction on the natural number k. This idea requires

some work, though. The sequent $p \wedge q \to r \vdash p \to (q \to r)$ has a proof

1	$p \wedge q \to r$	premise
2	p	assumption
3	q	assumption
4	$p \wedge q$	$\wedge$i 2, 3
5	r	$\to$e 1, 4
6	$q \to r$	$\to$i 3–5
7	$p \to (q \to r)$	$\to$i 2–6

but if we remove the last line or several of the last lines, we no longer have a proof as the outermost box does not get closed. We get a complete proof, though, by removing the last line and re-writing the assumption of the outermost box as a premise:

1	$p \wedge q \to r$	premise
2	p	premise
3	q	assumption
4	$p \wedge q$	$\wedge$i 2, 3
5	r	$\to$e 1, 4
6	$q \to r$	$\to$i 3–5

This is a proof of the sequent $p \wedge q \to r,\ p \vdash p \to r$. The induction hypothesis then ensures that $p \wedge q \to r,\ p \vDash p \to r$ holds. But then we can also reason that $p \wedge q \to r \vDash p \to (q \to r)$ holds as well – why?

Let's proceed with our proof by induction. We assume $M(k')$ for each $k' < k$ and we try to prove $M(k)$.

Base case: a one-line proof. If the proof has length 1 ($k = 1$), then it must be of the form

1 φ premise

since all other rules involve more than one line. This is the case when $n = 1$ and φ_1 and ψ equal φ, i.e. we are dealing with the sequent φ φ. Of course, since φ evaluates to T so does φ. Thus, φ φ holds as claimed.

Course-of-values inductive step: Let us assume that the proof of the sequent $\varphi_1, \varphi_2, \ldots, \varphi_n$ ψ has length k and that the statement we want to prove is true for all numbers less than k. Our proof has the following structure:

1 φ_1 premise
2 φ_2 premise
.
.
.
n φ_n premise
.
.
.
k ψ justification

There are two things we don't know at this point. First, what is happening in between those dots? Second, what was the last rule applied, i.e. what is the justification of the last line? The first uncertainty is of no concern; this is where mathematical induction demonstrates its power. The second lack of knowledge is where all the work sits. In this generality, there is simply no way of knowing which rule was applied last, so we need to consider all such rules in turn.